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The oscillator model for dissipative QED in an inhomogeneous dielectric

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Abstract
The Ullersma model for the damped harmonic oscillator is coupled to the quantized electromagnetic field. All material parameters and interaction strengths are allowed to depend on position. The ensuing Hamiltonian is expressed in terms of canonical fields, and diagonalized by performing a normal-mode expansion. The commutation relations of the diagonalizing operators are in agreement with the canonical commutation relations. For the proof we replace all sums of normal modes by complex integrals with the help of the residue theorem. The same technique helps us to explicitly calculate the quantum evolution of all canonical and electromagnetic fields. We identify the dielectric constant and the Green function of the wave equation for the electric field. Both functions are meromorphic in the complex frequency plane. The solution of the extended Ullersma model is in keeping with well-known phenomenological rules for setting up quantum electrodynamics in an absorptive and spatially inhomogeneous dielectric. To establish this fundamental justification, we subject the reservoir of independent harmonic oscillators to a continuum limit. The resonant frequencies of the reservoir are smeared out over the real axis. Consequently, the poles of both the dielectric constant and the Green function unite to form a branch cut. Performing an analytic continuation beyond this branch cut, we find that the long-time behaviour of the quantized electric field is completely determined by the sources of the reservoir. Through a Riemann–Lebesgue argument we demonstrate that the field itself tends to zero, whereas its quantum fluctuations stay alive. We argue that the last feature may have important consequences for application of entanglement and related processes in quantum devices.

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1. Introduction

More than a decade ago, Huttner and Barnett [1] published a valuable contribution to the complicated and long-established [2] subject of quantum electrodynamics in nonrelativistic macroscopic matter. Opting for a canonical setting, they carried out a fundamental derivation of the quantized electromagnetic field in an absorptive dielectric medium. To open up the possibility of working analytically, they avoided making direct contact with the atomic level. Instead, they described the properties of the dielectric with the help of harmonic oscillators. The influence of absorption was mimicked by coupling the dielectric to a continuum of harmonic oscillators. These reasonable simplifications gave rise to a so-called damped-polariton model that could be exactly solved. The quadratic Hamiltonian was diagonalized by means of Fano's method [3] and Fourier transformation.

The solution for the quantized electromagnetic field in an absorptive dielectric was welcomed by a large community. Phenomenological quantization schemes, the practical value of which was beyond dispute, could be put on a solid microscopic foundation. This possibility marked the beginning of a period of progress, in which our knowledge of QED in or near macroscopic matter was considerably broadened. For a lot of different optical media and experimental geometries, ranging from magnetic materials to beam splitters, quantization of the electromagnetic field was successfully carried out. At present, the macroscopic formulation of QED is well established. A few years ago, it was comprehensively reviewed [4, 5].

Over the years, it has been pointed out several times [6, 7] that the justification of Huttner and Barnett has one major shortcoming. The dielectric is assumed to be homogeneous in space, whereas in many experimental situations the electromagnetic field is substantially influenced by spatial inhomogeneities. One thus would like to extend the solution of the homogeneous damped-polariton model to the case in which all model parameters depend on position. This is a far from trivial enterprise, because the lack of translational invariance deprives us of the possibility to perform Fourier transformation. On the other hand, the quadratic character of the Hamiltonian remains intact, so there are no mathematical indications for a failure of Fano's procedure when passing over to the inhomogeneous case. Indeed, in two companion papers it is demonstrated that the inhomogeneous damped-polariton model can be solved as well. One can employ either Fano's procedure [8], or an alternative method that is based on Laplace transformation [9].

The extension of the work of Huttner and Barnett to dielectrics with spatial inhomogeneities does not complete the programme of underpinning phenomenological quantization rules. This judgement, which is the main motivation for undertaking the present study, is supported by several arguments. First, standard works on conservative QED [10–12] and classical fields [13, 14] insist on reducing the collection of canonical variables to an enumerable lot before commencing a canonical quantization procedure. In doing so for the inhomogeneous case, we are automatically led to a complete set of normal-mode functions, on the basis of which the electromagnetic fields can be expanded. We thus identify the functions that replace the plane waves of the homogeneous case. Moreover, we find the natural decomposition of the Green function [8, 9] belonging to the wave equation for the electric field.

Our second reservation with regard to the damped-polariton model concerns the loss mechanism. Instead of employing a continuum of oscillators right from the beginning, we should postpone the transition to the continuum as long as possible. This allows for a scrutiny of the mathematical origin of the absorptive behaviour. Also, we can completely clarify the relation between irreversibility of the dynamics and causality of the dielectric constant. Our third argument pertains to a technical observation on Fano's method in the presence of continua.

In solving for the diagonalizing operators one must introduce a formal contribution that is proportional to a Dirac delta function. We wish to improve upon this approach. To that end, we must refrain from employing distributions [3]. Fourth and last, we remark that already back in the 1960s quantum dissipation was studied with the help of harmonic oscillators [15–18]. It is important to find out whether the older oscillator models corroborate the predictions of the damped-polariton model.

We can meet the four suggestions advanced above by exchanging the continuum of the damped-polariton model for a reservoir containing a finite collection of independent harmonic oscillators. In essence, we add an electromagnetic sector to the Ullersma model [17] for a damped harmonic oscillator. As expected, we can still diagonalize the Hamiltonian of the extended Ullersma model. Eventually, we come up with a mathematical limit that restores the continuum.

In section 2, we specify the Hamiltonian of our model. In preparation of a swift canonical quantization procedure, we solve the classical evolution equations for the canonical fields by invoking a normal-mode expansion. Quantization of the dynamics happens in section 3. With each normal mode we associate a quantized harmonic oscillator. Next, making use of complex integration, we determine the evolution in time of all canonical fields. This permits us to identify the dielectric constant as well as the Green function. Subsequently, we turn on absorption by subjecting the reservoir to a continuum limit. We witness how poles unite and radically change the analytic structure of both the dielectric constant and the Green function. We derive the inhomogeneous counterpart of the solution that was obtained by Huttner and Barnett. Much attention is devoted to the long-time behaviour of the electric field. Section 4 contains a summary and guides the reader to the main results of our paper.

We close this introduction with some technical remarks. In performing partial integrations for canonical fields and related quantities, we tacitly assume that there are no contributions from the boundaries of the dielectric. Our aim to keep the treatment free from distributions forces us to utilize a considerable amount of basic function theory; all of the accompanying calculus is transferred to two appendices. In an attempt to present clear formulae, we omit spatial arguments whenever possible. The subscript $L(T)$ denotes that one should take the longitudinal (transverse) part of a vector field or tensor field. Last, we make use of rationalized mks units throughout.

2. Classical treatment

In this section, we derive the classical Hamiltonian of our model. We let us be guided by the Lagrange formalism, so our treatment bears a canonical character from the very outset. As usual, Hamilton's equations furnish the evolution laws for the canonical fields. In solving these, we rely on a normal-mode expansion. It appears that the dynamics is governed by an eigenvalue problem for a self-adjoint differential operator in three dimensions. For a spatially homogeneous dielectric the eigenvectors can be constructed from plane waves. To verify that the normal modes neatly decouple, we compute the Hamiltonian on the basis of the solutions for the canonical fields.

2.1. The model

The Ullersma model [17] describes the interaction between a single harmonic oscillator and a reservoir made up by an array of N independent harmonic oscillators. To fulfil our purposes, we assign to all $N + 1$ oscillators a spatial dependence. The corresponding position vector $\mathbf{r}$ covers a finite volume V. As a further extension of the Ullersma configuration we introduce an electromagnetic sector, denoting the electric field and magnetic field at time t as $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, respectively. In vacuum these fields give rise to the standard electromagnetic Lagrangian density

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\epsilon_0 \mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2. \tag{1}$$

The presence of a dielectric medium of volume V is taken into account by the privileged harmonic oscillator of the extended Ullersma model. It has mass density $\rho(\mathbf{r})$ and frequency $\omega_0(\mathbf{r})$. The field $\mathbf{Q}_0(\mathbf{r}, t)$ measures its displacement. The Lagrangian density of the dielectric reads

$$\mathcal{L}_D = \tfrac{1}{2}\rho\dot{\mathbf{Q}}_0^2 - \tfrac{1}{2}\rho\omega_0^2\mathbf{Q}_0^2. \tag{2}$$

After suitable scaling, all harmonic oscillators of the reservoir have a mass density of $\rho(\mathbf{r})$ as well. Their frequencies and displacement fields are equal to $\omega_n(\mathbf{r})$ and $\mathbf{Q}_n(\mathbf{r}, t)$, respectively. From (2) we see that

$$\mathcal{L}_R = \sum_{n=1}^{N}\left(\frac{1}{2}\rho\dot{\mathbf{Q}}_n^2 - \frac{1}{2}\rho\omega_n^2\mathbf{Q}_n^2\right) \tag{3}$$

is the Lagrangian density of the free reservoir.

In the electric-dipole approximation the electric field induces in the dielectric a polarization density $\mathcal{P} = -\alpha\mathbf{Q}_0$, where α is positive. In a strictly microscopic theory $-\alpha(\mathbf{r})$ would be a local electronic charge density. The interaction between the electromagnetic fields and the dielectric yields a contribution of $-\sigma_s\phi + \mathbf{j}_s \cdot \mathbf{A}$ to the Lagrangian density. In the absence of free charges and currents, one may substitute $\sigma_s = -\nabla\cdot\mathcal{P}$ and $\mathbf{j}_s = \dot{\mathcal{P}}$ for the sources. The electromagnetic potentials are determined by the definitions $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \nabla\times\mathbf{A}$, supplemented with the choice $\mathbf{A}_L = 0$, which is equivalent to the Coulomb gauge.

As long as it allows for energy exchange, the precise form of the interaction between the dielectric and the reservoir is not of physical interest to us. Introducing a coupling $\beta_n(\mathbf{r})$, we link the displacement field $\mathbf{Q}_0$ to the time derivative $\dot{\mathbf{Q}}_n$ rather than the field $\mathbf{Q}_n$ itself. This departure from the Ullersma model is common in studies of dissipative QED [1]. The two interactions of our model make the following contribution to the Lagrangian density:

$$\mathcal{L}_I = \alpha(\nabla\phi)\cdot\mathbf{Q}_0 - \alpha\mathbf{A}\cdot\dot{\mathbf{Q}}_0 - \sum_{n=1}^{N}\beta_n\mathbf{Q}_0\cdot\dot{\mathbf{Q}}_n. \tag{4}$$

We have added the total derivative $\nabla\cdot(\alpha\phi\mathbf{Q}_0)$, so that the nabla operator acts on the scalar potential.

As the Lagrangian $L = \int \mathrm{d}\mathbf{r}(\mathcal{L}_{\mathrm{EM}} + \mathcal{L}_D + \mathcal{L}_R + \mathcal{L}_I)$ does not depend on $\dot{\phi}$, one immediately finds $\nabla\phi = -(\alpha\mathbf{Q}_0)_L/\epsilon_0$ from the corresponding Euler–Lagrange equation. Owing to the Coulomb gauge, the character of the first term of (4) is purely electrostatic. The remaining Euler–Lagrange equations provide us with the inhomogeneous Maxwell equation for the vector potential, and the equations of motion for all $N + 1$ displacement fields.

Defining canonical momenta as

$$\mathbf{\Pi} = \frac{\delta L}{\delta\dot{\mathbf{A}}} \qquad \mathbf{P}_n = \frac{\delta L}{\delta\dot{\mathbf{Q}}_n} \tag{5}$$

for $0 \leqslant n \leqslant N$, we obtain the Hamiltonian

$$H = \int \mathrm{d}\mathbf{r}\left(\dot{\mathbf{A}}\cdot\mathbf{\Pi} + \sum_{n=0}^{N}\dot{\mathbf{Q}}_n\cdot\mathbf{P}_n\right) - L \tag{6}$$

as

$$H = \int \mathrm{d}\mathbf{r} \left[\frac{1}{2\epsilon_0}\mathbf{\Pi}^2 + \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 + \frac{\alpha^2}{2\rho}\mathbf{A}^2 + \frac{1}{2\epsilon_0}(\alpha \mathbf{Q}_0)_L^2 + \frac{1}{2\rho}\mathbf{P}_0^2 + \frac{1}{2}\rho\tilde{\omega}_0^2\mathbf{Q}_0^2 \right.$$
$$\left. + \sum_{n=1}^{N}\left(\frac{1}{2\rho}\mathbf{P}_n^2 + \frac{1}{2}\rho\omega_n^2\mathbf{Q}_n^2\right) + \frac{\alpha}{\rho}\mathbf{A}\cdot\mathbf{P}_0 + \sum_{n=1}^{N}\frac{\beta_n}{\rho}\mathbf{Q}_0\cdot\mathbf{P}_n \right]. \tag{7}$$

For the sum $\omega_0^2 + \sum_{n=1}^{N}\beta_n^2/\rho^2$ the abbreviation $\tilde{\omega}_0^2$ will be in use. The Hamiltonian of our model being available, we can start investigating the evolution of the canonical fields.

2.2. *Solution of Hamilton's equations*

Since we work in a finite volume, we may try to unravel the dynamics with the help of an enumerable set of independent normal modes. We shall see that each mode has a specific spatial structure and oscillates at a specific frequency. For that reason, the modes must be labelled by both a spatial index k and an integer l, which enumerates the mode frequencies $\Omega(k,l)$ for k fixed. The index j identifies the canonical fields with the auxiliary fields $\mathbf{Z}_j$ in the following manner:

$$\{\mathbf{\Pi}, \mathbf{A}, \mathbf{P}_0, \mathbf{Q}_0, \mathbf{P}_n, \mathbf{Q}_n\} = \{\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_{5n}, \mathbf{Z}_{6n}\} \tag{8}$$

with $n = 1, 2, 3, \ldots, N$. Now we can put forward our normal-mode expansion

$$\mathbf{Z}_j(\mathbf{r}, t) = \sum_{k,l} c(k,l)\mathbf{a}_j(k,l;\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\Omega(k,l)t} + \mathrm{cc}. \tag{9}$$

The frequencies $\Omega(k,l)$ are positive by definition. The coefficients $c(k,l)$ could be absorbed in the mode amplitudes $\mathbf{a}_j$, but this is inconvenient in view of the quantization procedure lying ahead of us. Because of the Coulomb gauge the amplitudes $\mathbf{a}_1$ and $\mathbf{a}_2$ are transverse vector fields.

Upon substituting (9) into Hamilton's equations and carefully evaluating functional derivatives, one arrives at

$$\begin{aligned}
&\mathrm{i}\Omega\mathbf{a}_1 = -\frac{1}{\mu_0}\triangle\mathbf{a}_2 + \left(\frac{\alpha^2}{\rho}\mathbf{a}_2\right)_T + \left(\frac{\alpha}{\rho}\mathbf{a}_3\right)_T && \mathrm{i}\Omega\mathbf{a}_2 = -\frac{1}{\epsilon_0}\mathbf{a}_1 \\
&\mathrm{i}\Omega\mathbf{a}_3 = \rho\tilde{\omega}_0^2\mathbf{a}_4 + \frac{\alpha}{\epsilon_0}(\alpha\mathbf{a}_4)_L + \sum_{m=1}^{N}\frac{\beta_m}{\rho}\mathbf{a}_{5m} && \mathrm{i}\Omega\mathbf{a}_4 = -\frac{\alpha}{\rho}\mathbf{a}_2 - \frac{1}{\rho}\mathbf{a}_3 \\
&\mathrm{i}\Omega\mathbf{a}_{5n} = \rho\omega_n^2\mathbf{a}_{6n} && \mathrm{i}\Omega\mathbf{a}_{6n} = -\frac{\beta_n}{\rho}\mathbf{a}_4 - \frac{1}{\rho}\mathbf{a}_{5n}
\end{aligned} \tag{10}$$

with $1 \leqslant n \leqslant N$. The arguments of $\mathbf{a}_j$ and Ω are identical to those appearing in (9). We emphasize that the model parameters α, β_n, ρ, $\tilde{\omega}_0$, and ω_n depend on position.

We set out to derive a wave equation for the amplitude $\mathbf{e}(k,l;\mathbf{r})$ of the electric field. The expansion

$$\mathbf{E}(\mathbf{r}, t) = \sum_{k,l} c(k,l)\mathbf{e}(k,l;\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\Omega(k,l)t} + \mathrm{cc} \tag{11}$$

should match with the definition of $\mathbf{E}$ in terms of the electromagnetic potentials. This brings us to the prescription

$$\epsilon_0\mathbf{e} = -\mathbf{a}_1 + (\alpha\mathbf{a}_4)_L\,. \tag{12}$$

The above relation enables us to present the solution of the algebraic part of (10) as

$$\begin{aligned}
\mathbf{a}_1 &= -\epsilon_0 \mathbf{e}_T & \mathbf{a}_2 &= -\frac{\mathrm{i}}{\Omega}\mathbf{e}_T \\
\mathbf{a}_3 &= \frac{\mathrm{i}\alpha}{\Omega}\mathbf{e}_T - \frac{\mathrm{i}\alpha\Omega}{h(\Omega)}\mathbf{e} & \mathbf{a}_4 &= \frac{\alpha}{\rho h(\Omega)}\mathbf{e} \\
\mathbf{a}_{5n} &= \frac{\alpha\beta_n\omega_n^2}{\rho h(\Omega)(\Omega^2-\omega_n^2)}\mathbf{e} & \mathbf{a}_{6n} &= \frac{\mathrm{i}\alpha\beta_n\Omega}{\rho^2 h(\Omega)(\Omega^2-\omega_n^2)}\mathbf{e}.
\end{aligned} \tag{13}$$

The new function $h(s)$ implicitly depends on position, and is given by

$$h(s) = s^2 - \tilde{\omega}_0^2 + \sum_{n=1}^{N} \frac{\beta_n^2\omega_n^2}{\rho^2\left(\omega_n^2 - s^2\right)}. \tag{14}$$

In the next section, the real-valued argument s will be replaced by a complex variable. A swift verification of (13) may take place through substitution into (10). Indeed all five algebraic equations are satisfied if (12) is used.

The first equation of (10), containing the Laplacian operator, has not been considered as yet. After substitution of (13) and employment of (12) it reduces to

$$c^2\nabla\times(\nabla\times\mathbf{e}) = \Omega^2\varepsilon(\Omega)\mathbf{e}. \tag{15}$$

This is the standard wave equation for the electric field, which appears in electromagnetic theory. The function

$$\varepsilon(s) = 1 - \frac{\alpha^2}{\epsilon_0\rho h(s)} \tag{16}$$

plays the role of dielectric constant. Its spatial dependence is made explicit in the following.

In (15) a linear differential operator is at work. It is given by

$$L(s;\mathbf{r})\mathbf{v}(\mathbf{r}) = -c^2\nabla\times[\nabla\times\mathbf{v}(\mathbf{r})] + s^2\varepsilon(s;\mathbf{r})\mathbf{v}(\mathbf{r}). \tag{17}$$

The parameter s is arbitrary, but still real-valued. The vector field $\mathbf{v}$ belongs to a Hilbert space of complex and square integrable functions, with scalar product defined as

$$\langle\mathbf{v}_1,\mathbf{v}_2\rangle = \int \mathrm{d}\mathbf{r}\,\mathbf{v}_1^*(\mathbf{r})\cdot\mathbf{v}_2(\mathbf{r}). \tag{18}$$

The integration is confined to the finite volume V. Since $L(s)$ is self-adjoint, its eigenvalues $\lambda(k,s)$ are real. Second, its eigenvectors $\mathbf{u}(k,s)$ possess the orthonormality property

$$\langle\mathbf{u}(k,s),\mathbf{u}(k',s)\rangle = \delta_{kk'}. \tag{19}$$

Obviously, (19) is not affected by degeneracy. Eigenspaces of higher dimension should be subjected to an orthonormalization procedure. Third, the eigenvectors make up a basis for the Hilbert space. We have a decomposition of the tensorial delta function at our disposal, given by

$$\sum_k \mathbf{u}^*(k,s;\mathbf{r}')\mathbf{u}(k,s;\mathbf{r}) = \boldsymbol{\delta}(\mathbf{r}'-\mathbf{r}). \tag{20}$$

The reality of the left-hand side follows from the invariance of the tensorial delta function under interchange of its arguments and indices. Last, as $L(s)$ is quadratic in the parameter s, the eigenvectors for s and $-s$ can be chosen identical to each other, so that the relation

$$\mathbf{u}(k,-s;\mathbf{r}) = \mathbf{u}(k,s;\mathbf{r}) \tag{21}$$

may be assumed.

Of course, for s equal to a mode frequency Ω, the eigenvalue problem

$$L(s;\mathbf{r})\mathbf{u}(k,s;\mathbf{r}) = \lambda(k,s)\mathbf{u}(k,s;\mathbf{r}) \tag{22}$$

should be in keeping with the wave equation (15). We therefore make the choice

$$\mathbf{e}(k,l;\mathbf{r}) = w(k,l)\mathbf{u}(k,\Omega(k,l);\mathbf{r}). \tag{23}$$

The weight w will be evaluated in due course. We furthermore require that the eigenvalue $\lambda(k,s)$ be equal to zero if s coincides with a mode frequency. In other words, by solving the equation

$$\lambda(k,s) = 0 \tag{24}$$

with s real-valued, we gather all mode frequencies. From (19) and (22) one infers

$$\lambda(k,s) = \langle \mathbf{u}(k,s), s^2\varepsilon(s)\mathbf{u}(k,s)\rangle - c^2\langle \nabla\times\mathbf{u}(k,s), \nabla\times\mathbf{u}(k,s)\rangle. \tag{25}$$

A partial integration has been performed. In appendix A we use (25) to take a closer look at the solutions of (24). From (21) we see that $\lambda(k,s)$ is even in s, so only pairs $s=\pm\Omega$ occur. The negative solutions generate the cc terms of (9).

In summary, Hamilton's equations have been solved by means of a normal-mode expansion. The spatial structure of the normal modes is described by the eigenvalue problem (22), whereas the mode frequencies are specified by the constraint (24). Now we should keep our promise to demonstrate that the modes do not interact with each other.

2.3. *Computation of the Hamiltonian*

We insert the normal-mode expansion (9) for all canonical fields into the Hamiltonian (7). Subsequently, we employ (13) and (23) to convert all amplitudes $\mathbf{a}_j(k,l)$ into eigenvectors $\mathbf{u}(k,\Omega)$. The arguments (k,l) of Ω are omitted. Likewise, by Ω' the mode frequency $\Omega(k',l')$ is meant. If we perform a partial integration, the wave equation (15) allows us to get rid of all differential operators. The longitudinal contributions of (7) can be eliminated via the relation

$$[\varepsilon(\Omega)\mathbf{u}(k,\Omega)]_L = 0 \tag{26}$$

which is a consequence of (15) and (23).

The Hamiltonian is equal to a sum over indices k,l,k',l'. We can interchange these to make all summands maximally symmetric. Then two classes of summands remain: those oscillating rapidly at frequency $\Omega+\Omega'$, and those oscillating slowly at frequency $\Omega-\Omega'$. For each summand of the first class there is a complex conjugate, which may be ignored in the following. For clarity, we remark that summands oscillating at frequency $-\Omega+\Omega'$ can be transferred to the second class. A simple interchange of summation indices suffices.

The summands of rapid oscillation will be treated first. All of these can be expressed in terms of h functions, owing to the relation

$$\sum_{n=1}^{N} \frac{\beta_n^2\omega_n^2(\omega_n^2 - ss')}{\rho^2(\omega_n^2 - s^2)(\omega_n^2 - s'^2)} = \frac{sh(s) + s'h(s')}{s+s'} - s^2 - s'^2 + ss' + \tilde{\omega}_0^2. \tag{27}$$

The sum on the left-hand side originates from (7). The dummies s and s' are set equal to Ω and Ω', respectively. Verification of (27) is straightforward after substitution of the definition (14) on the right-hand side. If we eliminate all h functions in favour of dielectric functions (16), the summands of rapid oscillation yield a form that vanishes after the use of the identity

$$\langle \mathbf{u}^*(k',\Omega'), \Omega'^2\varepsilon(\Omega')\mathbf{u}(k,\Omega) - \Omega^2\varepsilon(\Omega)\mathbf{u}(k,\Omega)\rangle = 0. \tag{28}$$

One proves (28) by taking the scalar product of $\mathbf{u}^*(k', \Omega')$ and the wave equation (15). Once again, partial integration is indispensable.

We have found that only the summands of slow oscillation contribute to the Hamiltonian. As long as Ω differs from Ω', these summands can be handled in the same manner as discussed above. There is only one difference. In order to finalize the calculation, one needs the identity

$$\langle \mathbf{u}(k', \Omega'), \Omega'^2 \varepsilon(\Omega')\mathbf{u}(k, \Omega) - \Omega^2 \varepsilon(\Omega)\mathbf{u}(k, \Omega)\rangle = 0 \tag{29}$$

instead of (28). The proofs of (28) and (29) follow the same path. Altogether, for $\Omega \neq \Omega'$ the summands of slow oscillation add up to zero as well.

The case $\Omega = \Omega'$ calls for employment of the relation

$$\sum_{n=1}^{N} \frac{\beta_n^2 \omega_n^2 \left(\omega_n^2 + s^2\right)}{\rho^2 \left(\omega_n^2 - s^2\right)^2} = \frac{\mathrm{d}s h(s)}{\mathrm{d}s} - 3s^2 + \tilde{\omega}_0^2. \tag{30}$$

Now elimination of h in favour of ε no longer produces a null result. We are led to

$$H = 2\epsilon_0 \sum_{k,l,k',l'}{}' c(k, l) c^*(k', l') w(k, l) w^*(k', l') \left\langle \mathbf{u}(k', s), \frac{\mathrm{d}s^2 \varepsilon(s)}{\mathrm{d}s^2} \mathbf{u}(k, s) \right\rangle_{s=\Omega(k,l)}. \tag{31}$$

Because of the constraint $\Omega = \Omega'$ the summation carries a prime. As the wave equation (15) is only valid for discrete values of Ω, it may not be differentiated with respect to this variable. Therefore, further manipulations of (31) must take place in a meticulous manner. From partial integration and (15) we learn

$$\left\langle \mathbf{u}(k', s), \frac{\mathrm{d}s^2 \varepsilon(s)}{\mathrm{d}s^2} \mathbf{u}(k, s) \right\rangle_{s=\Omega} = \left[\frac{\mathrm{d}}{\mathrm{d}s^2} \langle \mathbf{u}(k', s), L(s) \mathbf{u}(k, s)\rangle \right]_{s=\Omega} \tag{32}$$

with the condition $\Omega = \Omega'$ in force. The eigenvalue equation (22) holds true for all real values of s, so it may be utilized on the right-hand side of (32). Bearing in mind that the eigenvectors are orthonormal, and moreover, that at fixed k the correspondence between $\Omega(k, l)$ and l is one-to-one, we can finish our computation of the Hamiltonian. The diagonal form

$$H = 2\epsilon_0 \sum_{k,l} |c(k, l)|^2 |w(k, l)|^2 \left[\frac{\mathrm{d}\lambda(k, s)}{\mathrm{d}s^2} \right]_{s=\Omega(k,l)} \tag{33}$$

is found. The derivative is strictly positive, as shown in appendix A. In accordance with our expectations, the normal modes do not interact with each other.

We choose the weights as

$$w(k, l) = \epsilon_0^{-1/2} \Omega(k, l) \left\{ \left[\frac{\mathrm{d}\lambda(k, s)}{\mathrm{d}s} \right]_{s=\Omega(k,l)} \right\}^{-1/2} \tag{34}$$

so that (33) becomes

$$H = \sum_{k,l} \Omega(k, l) |c(k, l)|^2. \tag{35}$$

This is the Hamiltonian of an enumerable collection of independent harmonic oscillators. The quantity $c(k, l)$ is sometimes called the amplitude of the normal mode k, l [12]. With the normal-mode expansion (9) and the Hamiltonian (35) in hand, we fully understand how the classical dynamics of our model works.

2.4. Homogeneous dielectric

Analytic solution of the eigenvalue problem (22) will be impossible, except for special cases. One of these is a spatially homogeneous dielectric. The model parameters no longer depend on position. One can dispose of the parameter s figuring in the eigenvectors of (22). If one does not care for rotational or other spatial symmetries, then a simple solution for the eigenvectors is provided by plane waves.

A cube of side L serves as the volume V of the dielectric. The spatial index k splits up into a polarization index $\mu = 1, 2, 3$ and a wave vector $\mathbf{q} = 2\pi\mathbf{m}/L$, with m_j any integer. The eigenvectors $\mathbf{u}(k; \mathbf{r})$ are given by $L^{-3/2}\hat{\mathbf{o}}_\mu \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r})$, where $\hat{\mathbf{o}}_3$ equals $\hat{\mathbf{q}}$ and the set $\{\hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$ is orthonormal. In order to fulfil (20) the continuum limit $L \to \infty$ should be taken.

The solution of (22) and (25) is composed of a longitudinal part and a transverse part. The longitudinal and transverse mode frequencies obey the equations

$$\varepsilon(\Omega) = 0 \qquad \Omega^2\varepsilon(\Omega) = c^2\mathbf{q}^2. \tag{36}$$

The fact that the dielectric is invariant under rotations causes the degeneracy in (36).

3. Quantum treatment

In the previous section, we recognized that the classical evolution of our extended Ullersma model is controlled by an enumerable set of independent harmonic oscillators. For the quantization of each oscillator we resort to the old method of Dirac. The advantage is that the quantized Hamiltonian is delivered to us in diagonal shape. Since we work in a canonical setting throughout, we must ascertain that the canonical operators obey the canonical commutation relations. Completion of this job teaches us that discrete sums over normal modes can be transformed into complex integrals, if in the eigenvalue problem (22) the parameter s is replaced by a complex variable. The transition to complex integration is the key to disclosing the quantum evolution of the extended Ullersma model. For all Heisenberg operators, including the electric field and the electric displacement, integral representations are obtained. The dynamics is in agreement with Heisenberg's equations. To bring about dissipation in the dielectric, we make use of a continuum limit, which allots to the reservoir an uncountable number of degrees of freedom. Once the energy sink is activated, we can identify the causal dielectric function.

3.1. Dirac quantization

From here onwards, all capitals refer to quantum-mechanical operators, the Green function G, the operator L and the frequency Ω excepted. Following Dirac, we associate with each normal mode k, l of energy $\Omega(k,l)|c(k,l)|^2$ a harmonic oscillator. We define ladder operators through

$$c(k,l) \to \hbar^{1/2}C(k,l) \qquad c^*(k,l) \to \hbar^{1/2}C^\dagger(k,l) \tag{37}$$

and quantize by postulating

$$[C(k,l), C^\dagger(k',l')] = \delta_{kk'}\delta_{ll'} \qquad [C(k,l), C(k',l')] = 0. \tag{38}$$

Upon symmetrizing properly, we obtain the quantized counterpart of the Hamiltonian (35) as

$$H = \frac{\hbar}{2}\sum_{k,l}\Omega(k,l)[C^\dagger(k,l)C(k,l) + C(k,l)C^\dagger(k,l)]. \tag{39}$$

Each eigenvalue Ω is positive. From (9) and (37), we deduce that all canonical operators are represented by the expansion

$$\mathbf{Z}_j(\mathbf{r}, t) = \hbar^{1/2} \sum_{k,l} \mathbf{a}_j(k, l; \mathbf{r}) C(k, l)\, \mathrm{e}^{-\mathrm{i}\Omega(k,l)t} + \mathrm{hc}. \tag{40}$$

The amplitudes $\mathbf{a}_j$ are completely determined by the results established in the previous section. In calculating the quantized Hamiltonian, one may also depart from (7). After substitution of (40) into (7), one enters a rather lengthy road. The manipulations are essentially the same as in section 2.3, so there is no need for any comments. One indeed retrieves the diagonal form (39).

3.2. Canonical commutation relations

On our way to the swift quantization procedure (38), we let us be guided by the Lagrange formalism. Therefore, one may rightfully ask whether all is well with the canonical commutation relations. These read

$$\begin{aligned} &[\mathbf{A}(\mathbf{r}', t), \mathbf{\Pi}(\mathbf{r}, t)] = \mathrm{i}\hbar \boldsymbol{\delta}_T(\mathbf{r}' - \mathbf{r}) \\ &[\mathbf{Q}_0(\mathbf{r}', t), \mathbf{P}_0(\mathbf{r}, t)] = \mathrm{i}\hbar \boldsymbol{\delta}(\mathbf{r}' - \mathbf{r}) \\ &[\mathbf{Q}_{n'}(\mathbf{r}', t), \mathbf{P}_n(\mathbf{r}, t)] = \mathrm{i}\hbar \delta_{n'n} \boldsymbol{\delta}(\mathbf{r}' - \mathbf{r}). \end{aligned} \tag{41}$$

All other commutators of canonical fields equal zero. In the following we shall only verify the upper two commutators (41). For the other 19 cases new problems do not arise. Incidentally, for some commutators verification is trivial.

In virtue of (38) and (40) the upper two conditions (41) can be brought onto the form

$$\sum_{k,l} \frac{\epsilon_0}{\Omega(k, l)} \mathbf{e}_T^*(k, l; \mathbf{r}') \mathbf{e}_T(k, l; \mathbf{r}) + \mathrm{cc} = \boldsymbol{\delta}_T(\mathbf{r}' - \mathbf{r}) \tag{42}$$

$$\sum_{k,l} \frac{\alpha(\mathbf{r})\alpha(\mathbf{r}')\mathbf{e}^*(k, l; \mathbf{r}')}{\rho(\mathbf{r}')\Omega(k, l) h(\Omega(k, l); \mathbf{r}')} \left[\frac{\Omega^2(k, l)\mathbf{e}(k, l; \mathbf{r})}{h(\Omega(k, l); \mathbf{r})} - \mathbf{e}_T(k, l; \mathbf{r}) \right] + \mathrm{cc} = \boldsymbol{\delta}(\mathbf{r}' - \mathbf{r}). \tag{43}$$

Relations (23) and (34) suggest a recourse to the residue theorem. In moving to the complex plane we observe the rule

$$\mathbf{u}(k, s; \mathbf{r}) \rightarrow \mathbf{u}(k, z; \mathbf{r}) \qquad \mathbf{u}^*(k, s; \mathbf{r}) \rightarrow [\mathbf{u}(k, z^*; \mathbf{r})]^* \tag{44}$$

where $\mathbf{u}(k, z; \mathbf{r})$ is the solution of (21) and (22) with the replacement $s \rightarrow z$ carried out. In appendix A we argue that in (44) two entire functions of z figure. Moreover, we make plausible that the function $\lambda^{-1}(k, z)$ is meromorphic, and that its poles are given by $z = \pm\Omega(k, l)$, for k fixed.

Upon performing in (42) and (43) the transition to complex integration, we find the following three sufficient conditions:

$$\sum_k \int_{C_1} \frac{\mathrm{d}z}{\pi \mathrm{i}} \frac{z}{\lambda(k, z)} [\mathbf{u}_T(k, z^*; \mathbf{r}')]^* \mathbf{u}_T(k, z; \mathbf{r}) = \boldsymbol{\delta}_T(\mathbf{r}' - \mathbf{r}) \tag{45}$$

$$\sum_k \int_{C_1} \mathrm{d}z\, \frac{z}{h(z; \mathbf{r}')\lambda(k, z)} [\mathbf{u}(k, z^*; \mathbf{r}')]^* \mathbf{u}_T(k, z; \mathbf{r}) = 0 \tag{46}$$

$$\sum_k \int_{C_1} \frac{\mathrm{d}z}{\pi \mathrm{i}} \frac{z^3}{h(z; \mathbf{r})h(z; \mathbf{r}')\lambda(k, z)} [\mathbf{u}(k, z^*; \mathbf{r}')]^* \mathbf{u}(k, z; \mathbf{r}) = \frac{\epsilon_0 \rho(\mathbf{r})}{\alpha^2(\mathbf{r})} \boldsymbol{\delta}(\mathbf{r}' - \mathbf{r}). \tag{47}$$

The contour C_1 is composed of a set of circles running in counterclockwise sense (see figure B1 in appendix B). The lth circle encloses the pole on the positive real axis at $z = \Omega(k, l)$, with k fixed. Everywhere else in the interior of C_1 the integrands are analytic, because the circles can be chosen as small as we like. Hence, the use of the residue theorem in (45)–(47) indeed reproduces the sums over mode frequencies that are contained in (42) and (43).

The conditions (45)–(47) can be proved by performing a series of contour deformations. We defer this purely technical exercise to appendix B. Right now the reader should stay focused on the transition to complex integration, as practised above. We plan to exploit that skill in computing the time evolution of the canonical fields and other operators of physical interest. This is the goal of the next subsection.

3.3. Solution of the extended Ullersma model

To get a full picture of the dynamics of our quantum system, we have to specify the evolution of any set of initial canonical operators. In short, we have to establish the mapping between the times $t = 0$ and t for all canonical operators. To that end, we observe that any quantum operator can be written as a linear combination of the canonical operators at time zero. We apply this statement to $C^\dagger(k, l)$, and use (40) as well as (41) to make all coefficients explicit. We are led to the expansion

$$C^\dagger(k,l) = \frac{\mathrm{i}}{\hbar^{1/2}} \int \mathrm{d}\mathbf{r} \Bigg[\mathbf{a}_1(k,l;\mathbf{r}) \cdot \mathbf{A}(\mathbf{r},0) - \mathbf{a}_2(k,l;\mathbf{r}) \cdot \mathbf{\Pi}(\mathbf{r},0) + \mathbf{a}_3(k,l;\mathbf{r}) \cdot \mathbf{Q}_0(\mathbf{r},0) - \mathbf{a}_4(k,l;\mathbf{r}) \cdot \mathbf{P}_0(\mathbf{r},0) + \sum_{n=1}^{N} \mathbf{a}_{5n}(k,l;\mathbf{r}) \cdot \mathbf{Q}_n(\mathbf{r},0) - \sum_{n=1}^{N} \mathbf{a}_{6n}(k,l;\mathbf{r}) \cdot \mathbf{P}_n(\mathbf{r},0) \Bigg]. \tag{48}$$

One may wonder whether the representation (48) complies with the quantization prescription (38). By invoking (13), (23) and (41), one derives a set of consistency relations that is equivalent to (28), (32) and (34).

After substitution of (48) into (40) and the use of the normalization (34), we can express the vector potential as

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathrm{i}c^2}{\epsilon_0} \sum_{k,l} \int \mathrm{d}\mathbf{r}' \left\{ \mathrm{e}^{-\mathrm{i}st} \frac{\mathbf{u}_T(k,s;\mathbf{r})\mathbf{u}^*(k,s;\mathbf{r}') \cdot \mathbf{j}(s;\mathbf{r}')}{\mathrm{d}\lambda(k,s)/\mathrm{d}s} \right\}_{s=\Omega(k,l)} + \mathrm{hc}. \tag{49}$$

The source vector must be constructed from the initial canonical fields. One has

$$c^2\mathbf{j}(s;\mathbf{r}) = -\mathrm{i}\epsilon_0 s\mathbf{A}(\mathbf{r},0) + \mathbf{\Pi}(\mathbf{r},0) + [\alpha(\mathbf{r})\mathbf{Q}_0(\mathbf{r},0)]_T - \frac{s^2\alpha(\mathbf{r})\mathbf{Q}_0(\mathbf{r},0)}{h(s;\mathbf{r})} - \frac{\mathrm{i}s\alpha(\mathbf{r})\mathbf{P}_0(\mathbf{r},0)}{\rho(\mathbf{r})h(s;\mathbf{r})} + \sum_{n=1}^{N} \frac{s\alpha(\mathbf{r})\beta_n(\mathbf{r})}{\rho(\mathbf{r})h(s;\mathbf{r})\big[s^2 - \omega_n^2(\mathbf{r})\big]} \left[\mathrm{i}\omega_n^2(\mathbf{r})\mathbf{Q}_n(\mathbf{r},0) - \frac{s}{\rho(\mathbf{r})}\mathbf{P}_n(\mathbf{r},0) \right]. \tag{50}$$

The intermediate result (49) paves the way for the residue theorem, in a similar vein as before. The following result is reached:

$$\mathbf{A}(\mathbf{r},t) = \int \mathrm{d}\mathbf{r}' \int_{C_3} \frac{\mathrm{d}z}{2\pi\epsilon_0} \, \mathrm{e}^{-\mathrm{i}zt} \mathsf{G}_T(z;\mathbf{r},\mathbf{r}') \cdot \mathbf{j}(z;\mathbf{r}'). \tag{51}$$

The operation T refers to the argument $\mathbf{r}$. The contour C_3 encloses the real axis by means of two straight lines running from $+\infty + i\eta$ to $-\infty + i\eta$, and from $-\infty - i\eta$ to $+\infty - i\eta$, where η is infinitesimally positive. From (22) we see that the Green function, defined as

$$\mathsf{G}(z; \mathbf{r}, \mathbf{r}') = \sum_k \frac{c^2}{\lambda(k, z)} \mathbf{u}(k, z; \mathbf{r})[\mathbf{u}(k, z^*; \mathbf{r}')]^*, \tag{52}$$

satisfies the partial differential equation

$$c^{-2} L(z; \mathbf{r}) \mathsf{G}(z; \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{53}$$

where z must lie on the contour C_3. As shown in appendix B, the integrand of (51) has the same analytic structure as $\lambda^{-1}(k, z)$. From (14) it is clear that in (50) the factor of $\left[z^2 - \omega_n^2(\mathbf{r})\right]^{-1}$ does not give rise to any poles. Hence, in (51) the contour C_1 could be exchanged for C_3 without paying a price.

For the displacement field $\mathbf{Q}_0$ the discrete solution is given by

$$\mathbf{Q}_0(\mathbf{r}, t) = -\sum_{k,l} \int d\mathbf{r}' \left\{ \frac{c^2 s \alpha(\mathbf{r})\, e^{-ist}}{\epsilon_0 \rho(\mathbf{r}) h(s; \mathbf{r})} \frac{\mathbf{u}(k, s; \mathbf{r}) \mathbf{u}^*(k, s; \mathbf{r}') \cdot \mathbf{j}(s; \mathbf{r}')}{d\lambda(k, s)/ds} \right\}_{s=\Omega(k,l)} + \text{hc}. \tag{54}$$

The transition to complex integration is immediate, provided that C_1 is chosen as contour. In deforming the latter to C_3, attention must be paid to denominators in which a double factor of h figures. Correction terms show up, which can be found by eliminating the nasty denominators with the help of the auxiliary function f, defined in appendix B. One should proceed along the same lines as for the proof of (47). This results in

$$\begin{aligned} \mathbf{Q}_0(\mathbf{r}, t) = &-\int d\mathbf{r}' \int_{C_3} \frac{dz}{2\pi i \epsilon_0} \frac{z \alpha(\mathbf{r})\, e^{-izt}}{\rho(\mathbf{r}) h(z; \mathbf{r})} \mathsf{G}(z; \mathbf{r}, \mathbf{r}') \cdot \mathbf{j}(z; \mathbf{r}') \\ &- \int_{C_3} \frac{dz}{2\pi i} \frac{e^{-izt}}{z \alpha(\mathbf{r})} \{c^2 \mathbf{j}(z; \mathbf{r}) + i\epsilon_0 z \mathbf{A}(\mathbf{r}, 0) - \mathbf{\Pi}(\mathbf{r}, 0) - [\alpha(\mathbf{r}) \mathbf{Q}_0(\mathbf{r}, 0)]_T\}. \end{aligned} \tag{55}$$

The local character of the correction term stems from application of the completeness relation (20).

The displacement field of the reservoir contains two local correction terms. The solution comes out as

$$\begin{aligned} \mathbf{Q}_n(\mathbf{r}, t) = &-\int d\mathbf{r}' \int_{C_3} \frac{dz}{2\pi \epsilon_0} \frac{z^2 \alpha(\mathbf{r}) \beta_n(\mathbf{r})\, e^{-izt}}{\rho^2(\mathbf{r}) h(z; \mathbf{r}) \left[z^2 - \omega_n^2(\mathbf{r})\right]} \mathsf{G}(z; \mathbf{r}, \mathbf{r}') \cdot \mathbf{j}(z; \mathbf{r}') \\ &- \int_{C_3} \frac{dz}{2\pi} \frac{\beta_n(\mathbf{r})\, e^{-izt}}{\alpha(\mathbf{r}) \rho(\mathbf{r}) \left[z^2 - \omega_n^2(\mathbf{r})\right]} \{c^2 \mathbf{j}(z; \mathbf{r}) + i\epsilon_0 z \mathbf{A}(\mathbf{r}, 0) - \mathbf{\Pi}(\mathbf{r}, 0) \\ &- [\alpha(\mathbf{r}) \mathbf{Q}_0(\mathbf{r}, 0)]_T\} + \int_{C_3} \frac{dz}{2\pi i} \frac{e^{-izt}}{\left[z^2 - \omega_n^2(\mathbf{r})\right]} \left[z \mathbf{Q}_n(\mathbf{r}, 0) + \frac{i}{\rho(\mathbf{r})} \mathbf{P}_n(\mathbf{r}, 0) \right] \end{aligned} \tag{56}$$

where one has $n = 1, 2, 3, \ldots, N$. The last correction term covers the special case of $\alpha = \beta_n = 0$. The fields $\mathbf{A}, \mathbf{Q}_0$ and $\mathbf{Q}_n$ provide us with the following solutions for the canonical momenta:

$$\mathbf{\Pi} = \epsilon_0 \dot{\mathbf{A}} \qquad \mathbf{P}_0 = \rho \dot{\mathbf{Q}}_0 - \alpha \mathbf{A} \qquad \mathbf{P}_n = \rho \dot{\mathbf{Q}}_n - \beta_n \mathbf{Q}_0 \tag{57}$$

with $n = 1, 2, 3, \ldots, N$. These expressions are obtained from the definitions (5).

All solutions for the canonical operators reproduce the initial condition if the choice $t = 0$ is made. This can be verified on the basis of the identities underlying the canonical commutation relations, such as (45) and (47). The transformation $z \to -z^*$ shows that the condition of self-adjointness is respected as well. One needs the symmetry property

$$[\mathsf{G}(-z^*; \mathbf{r}, \mathbf{r}')]^* = \mathsf{G}(z; \mathbf{r}, \mathbf{r}') \tag{58}$$

where z belongs to C_3. For the proof one combines (52) with the reciprocity relation

$$\mathsf{G}(z; \mathbf{r}', \mathbf{r})_{ji} = \mathsf{G}(z; \mathbf{r}, \mathbf{r}')_{ij}. \tag{59}$$

This last result follows by taking the scalar product of (53) with $\mathsf{G}(z; \mathbf{r}, \mathbf{r}'')$, and performing a partial integration.

All canonical fields obey the Heisenberg equation $\mathrm{i}\hbar\dot{\mathbf{Z}}_j = [\mathbf{Z}_j, H]$. To demonstrate this, one first calculates the commutator with H on the basis of the canonical commutation relations. Subsequently, one employs (57) to eliminate all canonical momenta. Last, one substitutes the integral solutions (51), (55) and (56). Contour deformations are not required; the use of the partial differential equation (53) is sufficient.

To compute the time evolution of the electric field we repeat the programme described above for (11). This brings us to

$$\mathbf{E}(\mathbf{r}, t) = -\int \mathrm{d}\mathbf{r}' \int_{C_3} \frac{\mathrm{d}z}{2\pi \mathrm{i}\epsilon_0} z\, \mathrm{e}^{-\mathrm{i}zt} \mathsf{G}(z; \mathbf{r}, \mathbf{r}') \cdot \mathbf{j}(z; \mathbf{r}'). \tag{60}$$

Combination of (55) and (60) gives for the electric displacement

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \mathbf{E}(\mathbf{r}, t) - \alpha(\mathbf{r})\mathbf{Q}_0(\mathbf{r}, t) = \mathbf{D}(\mathbf{r}, 0) + \int_{C_3} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}zt}}{z} c^2 \mathbf{j}(z; \mathbf{r}) \\ - \int \mathrm{d}\mathbf{r}' \int_{C_3} \frac{\mathrm{d}z}{2\pi \mathrm{i}} z\varepsilon(z; \mathbf{r})\, \mathrm{e}^{-\mathrm{i}zt} \mathsf{G}(z; \mathbf{r}, \mathbf{r}') \cdot \mathbf{j}(z; \mathbf{r}'). \end{aligned} \tag{61}$$

From (60) and (61), we conclude that one may indeed regard $\varepsilon(z; \mathbf{r})$ as a dielectric function. With the help of (17) one can clarify the status of the local contribution to the electric displacement. It guarantees that the field $\mathbf{D}_L(\mathbf{r}, t)$ does not exist, as prescribed by Maxwell's equations. Indeed, if one takes the divergence of (61) and inserts (53), the right-hand side reduces to the divergence of $\mathbf{D}(\mathbf{r}, 0)$, which equals zero.

In [1] the electric field and electric displacement were calculated for the case of a spatially homogeneous dielectric, coupled to an uncountable number of harmonic oscillators. We can extend these results to the case of an inhomogeneous dielectric by considering (60) and (61) for a reservoir, the eigenfrequencies of which make up a dense set. This will be done in the next subsection.

3.4. Continuum limit

For finite N the solution of the extended Ullersma model describes reversible dynamics. Therefore, the energy exchange between dielectric and reservoir is everlasting. On the other hand, for many experiments on optical properties of dielectrics, the absorption of photons is omnipresent. Hence, there is still a gap between the solutions of the previous subsection and experiment. Our dielectric medium does exhibit damping phenomena if we manage to create an irreversible energy flow into the reservoir. For that purpose, the reservoir should possess an uncountable number of degrees of freedom. Such a continuum comes into existence upon defining

$$\begin{aligned} &\Lambda^{1/2}\mathbf{Q}_n(\mathbf{r}, t) = \mathbf{Q}(n/\Lambda; \mathbf{r}, t) \qquad && \Lambda^{1/2}\mathbf{P}_n(\mathbf{r}, t) = \mathbf{P}(n/\Lambda; \mathbf{r}, t) \\ &\Lambda^{1/2}\beta_n(\mathbf{r}) = \beta(n/\Lambda; \mathbf{r}) && \omega_n(\mathbf{r}) = n/\Lambda \end{aligned} \tag{62}$$

for $n = 1, 2, 3, \ldots, N$, and taking the limit $\Lambda, N \to \infty$. Since n becomes arbitrarily large, the ratio n/Λ may be treated as a continuous variable ω. Note that the spatial dependence of ω_n does not leave any traces. The subscript c indicates that the continuum limit is taken.

We assume that for all nonnegative ω the function $\beta(\omega)$ is regular and smooth. If it decays faster than $1/\sqrt{\omega}$ for large ω, then the definition below (7) implies that the continuum limit of $\tilde{\omega}_0$ exists. The analytic properties of the function

$$h_c(z) = z^2 - \tilde{\omega}_{0,c}^2 + \int_0^\infty \mathrm{d}\omega \, \frac{\omega^2\beta^2(\omega)}{\rho^2(\omega^2 - z^2)} \tag{63}$$

radically differ from those of $h(z)$, given in (14). The zeros of $h(z)$ have united so as to generate a branch cut on the real axis. The same mechanism is witnessed for the dielectric function $\varepsilon_c(z)$ and the eigenvalue $\lambda_c(k, z)$. In appendix A we demonstrate that all of the afore-mentioned functions are analytic and nonzero outside the real axis, that is to say, both on the branch in the upper half plane and on the branch in the lower half plane. We assume that the analytic properties of the eigenvector $\mathbf{u}(k, z)$ are not affected by the continuum limit.

Now we are well prepared to find out how the solution of the extended Ullersma model, which was derived in the previous subsection, behaves under the continuum limit. We focus on the electric field, specified in (60). Treatment of the electric displacement and other fields goes by the same methodology. As mentioned earlier, the contour C_3 is composed of two straight lines enclosing the real axis. We parametrize C_3 as $\int \mathrm{d}z\, f(z) = \int_{-\infty}^{\infty} \mathrm{d}\omega[f(\omega - \mathrm{i}\eta) - f(\omega + \mathrm{i}\eta)]$, where η is infinitesimally positive. Next, we simplify the integrand with the help of the symmetry $[\mathsf{G}(z^*; \mathbf{r}, \mathbf{r}')]^* = \mathsf{G}(z; \mathbf{r}, \mathbf{r}')$. Application of rule (62) then leads to

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) = &\int \mathrm{d}\mathbf{r}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi\epsilon_0}\, \mathrm{e}^{-\mathrm{i}\omega t} \omega\, \mathrm{Im}[\mathsf{G}_c(\omega + \mathrm{i}\eta; \mathbf{r}, \mathbf{r}')] \cdot \mathbf{j}_{\mathrm{EM}}(\omega; \mathbf{r}') \\
&- \int \mathrm{d}\mathbf{r}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi \mathrm{i}\epsilon_0}\, \mathrm{e}^{-\mathrm{i}\omega t} \omega\alpha(\mathbf{r}')\, \mathrm{Im}\left[\frac{\mathsf{G}_c(\omega + \mathrm{i}\eta; \mathbf{r}, \mathbf{r}')}{h_c(\omega + \mathrm{i}\eta; \mathbf{r}')}\right] \cdot \mathbf{j}_D(\omega; \mathbf{r}') \\
&+ \int \mathrm{d}\mathbf{r}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi\epsilon_0} \int_0^\infty \mathrm{d}\omega'\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{\omega^2\alpha(\mathbf{r}')\beta(\omega'; \mathbf{r}')}{\rho(\mathbf{r}')} \\
&\times \mathrm{Im}\left\{\frac{\mathsf{G}_c(\omega + \mathrm{i}\eta; \mathbf{r}, \mathbf{r}')}{h_c(\omega + \mathrm{i}\eta; \mathbf{r}')[(\omega + \mathrm{i}\eta)^2 - \omega'^2]}\right\} \cdot \mathbf{j}_R(\omega, \omega'; \mathbf{r}').
\end{aligned} \tag{64}$$

The continuum Green function $\mathsf{G}_c(\omega + \mathrm{i}\eta; \mathbf{r}, \mathbf{r}')$ is determined by (52), with the replacements $z \to \omega + \mathrm{i}\eta$ and $\lambda \to \lambda_c$ carried out. The new source vectors are given by

$$\begin{aligned}
c^2\mathbf{j}_{\mathrm{EM}}(\omega; \mathbf{r}) &= -\mathrm{i}\epsilon_0\omega\mathbf{A}(\mathbf{r}, 0) + \mathbf{\Pi}(\mathbf{r}, 0) + [\alpha(\mathbf{r})\mathbf{Q}_0(\mathbf{r}, 0)]_T \\
c^2\mathbf{j}_D(\omega; \mathbf{r}) &= \mathrm{i}\omega^2\mathbf{Q}_0(\mathbf{r}, 0) - \frac{\omega}{\rho(\mathbf{r})}\mathbf{P}_0(\mathbf{r}, 0) \\
c^2\mathbf{j}_R(\omega, \omega'; \mathbf{r}) &= \mathrm{i}\omega'^2\mathbf{Q}(\omega'; \mathbf{r}, 0) - \frac{\omega}{\rho(\mathbf{r})}\mathbf{P}(\omega'; \mathbf{r}, 0).
\end{aligned} \tag{65}$$

In [9] the above solution is derived on the basis of Laplace transformation. In that paper, the reservoir contains a continuum of oscillators right from the beginning.

By taking the continuum limit as demonstrated above, we fully extend the results of [1] to the case of an inhomogeneous dielectric. However, there exists an alternative manner to implement the continuum limit. Instead of keeping the upper and lower parts of C_3 together, one can decide to sever these parts from each other. The ensuing representations are useful to analyse how fields behave for long times. We keep the time strictly positive and focus again on the electric field. For some integrands figuring in (60) it is wise to perform first the substitution $z/[f(z)] = [z^2 - f(z)]/[zf(z)] + 1/z$, where $f(z)$ stands for $\lambda_c(k, z)$ or $h_c(z)$. One then isolates the term $1/z$, which decays slowly as $|z|$ becomes large. The convergence of the remaining integrand improves.

The lower part of C_3 can be closed by means of the arc $z = R\exp(\mathrm{i}\phi)$, with $-\pi \leqslant \phi \leqslant 0$. Owing to the choice $t > 0$, the exponential factor $\exp(-\mathrm{i}zt)$ causes the integrand to shrink to zero on the arc. In the interior of the closed contour the integrand is analytic, so the lower part of C_3 does not make any contribution. The upper part of C_3 yields an integral over all real frequencies ω, which is convergent in virtue of the above substitution. Of course, for the term $1/z$ one cannot treat the upper and lower parts of C_3 separately; instead, the residue theorem offers a way out.

If we execute the above instructions for the electric field, then the following expression emerges:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},0) &+ \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}\epsilon_0}\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{\alpha(\mathbf{r})[h_c(\omega+\mathrm{i}\eta;\mathbf{r})-\omega^2]}{(\omega+\mathrm{i}\eta)h_c(\omega+\mathrm{i}\eta;\mathbf{r})}\mathbf{Q}_0(\mathbf{r},0)\\ &+\int \mathrm{d}\mathbf{r}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}\epsilon_0}\, \mathrm{e}^{-\mathrm{i}\omega t}\left[\omega \mathsf{G}_c(\omega+\mathrm{i}\eta;\mathbf{r},\mathbf{r}') - \frac{c^2}{\omega+\mathrm{i}\eta}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\right]\\ &\cdot\left[\mathbf{j}_{\mathrm{EM}}(\omega;\mathbf{r}') - \frac{\omega^2\alpha(\mathbf{r}')\mathbf{Q}_0(\mathbf{r}',0)}{c^2h_c(\omega+\mathrm{i}\eta;\mathbf{r}')}\right]\\ &+\int \mathrm{d}\mathbf{r}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}\epsilon_0}\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{\omega^2\alpha(\mathbf{r}')}{\rho(\mathbf{r}')h_c(\omega+\mathrm{i}\eta;\mathbf{r}')}\mathsf{G}_c(\omega+\mathrm{i}\eta;\mathbf{r},\mathbf{r}')\\ &\cdot\left[-\frac{\mathrm{i}}{c^2}\mathbf{P}_0(\mathbf{r}',0) + \int_0^{\infty}\mathrm{d}\omega' \frac{\beta(\omega';\mathbf{r}')}{(\omega+\mathrm{i}\eta)^2-\omega'^2}\mathbf{j}_R(\omega,\omega';\mathbf{r}')\right]. \qquad (66)\end{aligned}$$

As before, η is infinitesimally positive. In computing $h_c(\omega+\mathrm{i}\eta;\mathbf{r})$, the upper branch of the function (63) must be used. Consequently, each integrand of (66) is analytic in the upper half of the complex ω plane. For $t = 0$, one may then replace the contour by an arc $\omega = R\exp(\mathrm{i}\phi)$, with $0 \leqslant \phi \leqslant \pi$. On account of (14), (20) and (22), each integrand decays as $1/R^2$ or faster, so there are no integrals surviving the choice $t = 0$. In the continuum limit, the initial condition for the electric field is still satisfied.

To uncover irreversible behaviour, we have to analyse (66) for large times. If the symmetry relation

$$[\beta^2(-z^*)]^* = \beta^2(z) \qquad (67)$$

holds true, and $\beta(z)$ is well behaved at the origin, then the analytic continuation of $\varepsilon_c(\omega+\mathrm{i}\eta)$ and $\lambda_c(k,\omega+\mathrm{i}\eta)$ can be effectuated. Thus, the infinitesimal increment $\mathrm{i}\eta$ of the argument ω becomes redundant. From (16) and the continuation of $h_c(\omega+\mathrm{i}\eta)$ we obtain the symmetry relation $[\varepsilon_c(-\omega^*)]^* = \varepsilon_c(\omega)$, which is in accordance with classical theory of the dielectric constant [19]. All this is shown in appendix A. There we also prove that $\varepsilon_c(\omega)$ is analytic and different from zero for all $\mathrm{Im}\,\omega \geqslant 0$. Incidentally, if the symmetry (67) is absent, indeed circumstances exist under which the process of analytic continuation fails [20].

Now everything is ready to shift in (66) the contour into the lower half plane. The poles at $\omega = -\mathrm{i}\eta$ yield residues that erase the initial condition for the electric field. We adapt γ such that all integrands, accompanying $\mathbf{A}$, $\boldsymbol{\Pi}$, $\mathbf{Q}_0$, or $\mathbf{P}_0$, are analytic in the strip $-\gamma \leqslant \mathrm{Im}\,\omega \leqslant 0$. Then one may integrate along the line $\mathrm{Im}\,\omega = -\gamma$ instead of the real axis. The parameter γ surely differs from zero, otherwise the above process of analytic continuation would fail. The exponential $\exp(-\mathrm{i}\omega t)$ produces a factor of $\exp(-\gamma t)$, which induces absorption in the dielectric, as desired. By comparing (60) and (66) one can trace the origin of the irreversible behaviour. It resides in the fact that, as a result of the continuum limit, the analytic properties of $h(z)$ undergo a radical change.

The use of (16) in (66) gives a dielectric function $\varepsilon_c(\omega)$ that is analytic in the upper half plane. Therefore, irreversibility for $t \to \infty$ is coupled to causality for the dielectric function. Indeed, if one takes the continuum limit for t negative, one has to operate on the lower branch of (63). Then the dielectric function is analytic in the lower half of the complex ω plane. One encounters the combination of irreversibility for $t \to -\infty$ and anti-causality for the dielectric function. Of course, these conclusions do not depend on (67). In the absence of this symmetry one may resort to arguments of Riemann–Lebesgue type.

The terms of (66) that contain the canonical fields of the reservoir have not yet been considered. The corresponding integrand has poles at $\omega = \pm\omega' - \mathrm{i}\eta$, the residues of which generate oscillating contributions. Upon using the symmetry relations for G_c and h_c, we arrive at the following asymptotic expression for the electric field, valid for large times:

$$\mathbf{E}(\mathbf{r}, t) \sim -\int \mathrm{d}\mathbf{r}' \int_0^\infty \mathrm{d}\omega \exp(-\mathrm{i}\omega t - \mathrm{i}\phi) \left\{ \frac{\omega\rho(\mathbf{r}') \,\mathrm{Im}[\varepsilon_c(\omega; \mathbf{r}')]}{2\pi\epsilon_0} \right\}^{1/2} \\ \times \mathsf{G}_c(\omega; \mathbf{r}, \mathbf{r}') \cdot \mathbf{j}_R(\omega, \omega; \mathbf{r}') + \mathrm{hc}. \tag{68}$$

The phase $\phi = \arg(h_c)$ results from eliminating the coupling parameters α and β in favour of the dielectric function. To do this, the usual distributional calculus should be applied to (63). The imaginary part of the integrand on the right-hand side of (63) is proportional to a Dirac delta function.

By virtue of a Riemann–Lebesgue argument, the expectation value of the integral of (68) vanishes for large times. Hence, the same is true for the electric-field operator $\mathbf{E}(\mathbf{r}, t)$. Still, the oscillatory contributions of the reservoir qualitatively differ from those exhibiting exponential damping. The oscillations bring about quantum fluctuations that do not die out for large times. To make this explicit, we model the initial correlations in the reservoir as

$$\langle \mathbf{P}(\omega; \mathbf{r}, 0)\mathbf{P}(\omega'; \mathbf{r}', 0)\rangle = \tfrac{1}{3}\langle \mathbf{P}^2(\omega; \mathbf{r}, 0)\rangle \delta(\omega - \omega')\boldsymbol{\delta}(\mathbf{r} - \mathbf{r}') \tag{69}$$

where the last $\boldsymbol{\delta}$ function is a tensor. The brackets indicate that an expectation value is taken. For the auto-correlations of $\mathbf{Q}(\omega; \mathbf{r}, 0)$ the above model is assumed as well. Cross-correlations of $\mathbf{P}(\omega; \mathbf{r}, 0)$ and $\mathbf{Q}(\omega; \mathbf{r}, 0)$ are discarded. From the electric field we construct a quadratic form, take the expectation value, and process the result with the help of (68) and (69). Utilizing again a Riemann–Lebesgue argument and introducing the energy density of the reservoir as

$$\mathcal{H}_R(\omega; \mathbf{r}) = \tfrac{1}{2}\rho^{-1}(\mathbf{r})\mathbf{P}^2(\omega; \mathbf{r}, 0) + \tfrac{1}{2}\omega^2\rho(\mathbf{r})\mathbf{Q}^2(\omega; \mathbf{r}, 0) \tag{70}$$

we find the asymptotic result

$$\lim_{t\to\infty} \langle \mathbf{E}(\mathbf{r}, t)\mathbf{E}(\mathbf{r}', t)\rangle = \int \mathrm{d}\mathbf{r}'' \int_0^\infty \frac{\mathrm{d}\omega}{3\pi\epsilon_0 c^4} \omega^3 \mathsf{G}_c(\omega; \mathbf{r}, \mathbf{r}'') \\ \cdot \mathsf{G}_c^*(\omega; \mathbf{r}'', \mathbf{r}') \,\mathrm{Im}[\varepsilon_c(\omega; \mathbf{r}'')]\langle \mathcal{H}_R(\omega; \mathbf{r}'')\rangle + \mathrm{cc}. \tag{71}$$

In the continuum limit any expectation value of the electric field will eventually decay to zero, but the quantum fluctuations stay alive. They are fuelled by the energy that is available in the reservoir.

4. Summary and conclusion

The quantization of the electromagnetic field in the presence of nonrelativistic macroscopic matter constitutes a problem of formidable magnitude. In principle, any serious treatment should start from the minimal-coupling Hamiltonian that takes into account all electromagnetic interactions at the atomic level. It needs no argument that such a rigorous approach is scarcely possible, unless one is prepared to deploy heavy numerical means as soon as it comes to predicting experimental findings. However, one then misses the opportunity to see what kind of mathematical mechanisms are at work behind such processes as spontaneous emission or scattering of photons. Therefore, already during the early years of the subject people were looking for shortcuts so as to obtain concise and transparent theories [2]. Nowadays a generally accepted quantization scheme of phenomenological nature is available.

Because of their experimental relevance, rules for QED in absorptive matter deserve to be put on a firm foundation. This idea was pursued by Huttner and Barnett [1]. Sacrificing the direct contact with the atomic level, they solved a so-called damped-polariton model. Their microscopic expression for the quantized electromagnetic field in an absorptive dielectric initiated a lot of activity on the construction of phenomenological quantization rules. Despite this progress, to experimentalists a very important issue is still pending, namely, the microscopic quantization in a dielectric with both spatial inhomogeneities and losses. In addition to that, it is unclear how the damped-polariton model relates to the old oscillator models for dissipative quantum dynamics that were proposed in the 1960s [15–18].

Motivated by the last remarks, we exchange in this work the continuum of the damped-polariton model for a reservoir that consists of a finite number of independent harmonic oscillators. We thus consider the Ullersma configuration [17], extended with an electromagnetic sector. For all parameters, a spatial dependence is admitted, so that the oscillator model at hand is capable of describing spatial inhomogeneities. To give the reader the chance to make a direct comparison between the quantum and the classical evolution of the model, we first solve Hamilton's equations for the classical canonical fields. The latter are expressed as sums of independent normal modes. By associating with each normal mode a harmonic oscillator, we can invoke Dirac's method for the quantization of the dynamics. The corresponding ladder operators can be employed to cast the Hamiltonian into diagonal form. Hence, the time evolution of all fields can be made explicit. Having derived the solution of the extended Ullersma model, we can identify the inhomogeneous dielectric function. Also, we can meet one of our prime objectives, the generalization of the results of Huttner and Barnett [1] to the case of an inhomogeneous dielectric. This job is completed by subjecting our discrete reservoir to a continuum limit, which makes the coupling parameter of each separate oscillator infinitely weak, and at the same time converts the collection of eigenfrequencies into a dense set.

The discrete character of the Ullersma reservoir brings us important advantages. We can completely avoid the mathematical complexities that accompany the use of distributions. For example, we need not adapt the solution of the evolution equations through addition of an unknown term, containing a delta function [1, 3, 8]. To derive the results of this paper only simple technical tools are required. We apply linear algebra to prove orthogonality and completeness for the eigenvectors of operator L, which governs the spatial structure of the normal modes. Hence, these eigenvectors are the natural candidates for composing the Green function, as becomes manifest when computing the solutions for the canonical fields. The other standard ingredient we call in is the residue theorem. It allows us to reduce all discrete sums of normal modes to complex integrals. The ensuing contour C_1 is rather awkward, as it is composed of separate circles around poles on the real axis. Fortunately, C_1 gives way to frequency integrals upon performing contour deformation. In doing so, we generate correction terms of local nature, in which the Green function no longer figures. These terms can be interpreted as Langevin noise operators [1]. The local correction to the electric displacement ensures that the divergence of the latter equals zero, in accordance with Maxwell's equations. The easy access to the mathematical structure of our solutions enables us to scrutinize their behaviour under the continuum limit. The function h plays a central role. Its poles gradually cover the real axis, and finally give rise to a branch cut. The dielectric function is analytic on both sides of the cut, but the sign of the time t determines on which branch the fields differ from zero. This leads to the conclusion that (anti)causality of the dielectric function is intimately linked to irreversibility of the dynamics for t approaching (minus) infinity.

A further advantage offered by the Ullersma reservoir is the connection with recent work on decoherence and entanglement. Several papers describe how a reservoir of the Ullersma type can bring about decoherence of quantum superpositions and entanglement between qubits [21–23]. Results on these processes are valuable with an eye to quantum computing and other future applications. Inevitably, in any real device electromagnetic fields will participate in the interactions. Therefore, it would be worthwhile to study the reservoir-induced quantum phenomena in the presence of an electromagnetic sector. One would like to assess the influence of the quantum noise that is produced by the electromagnetic field. As we saw in this work, the energy density of the reservoir fuels the quantum fluctuations of the electric field. They outlive any decoherence or entanglement. The possible fragility of these quantum processes can surely be investigated with the aid of the extended Ullersma model. In particular the influence of temperature must be critically regarded.

Appendix A. Results for *h* and related functions

In the main text, we exploited some analytic properties of the functions $h(z; \mathbf{r})$, $\varepsilon(z; \mathbf{r})$ and $\lambda(k, z)$. For the first two functions, all properties can be proved by means of simple techniques, as we shall discuss below. When we turn to $\lambda(k, z)$, however, the going becomes heavier, due to the fact that the partial differential equation (22) intervenes. We shall need a few plausibility arguments in order to make progress. A rigorous mathematical investigation of (22) lies outside the scope of this paper.

The function $h(z; \mathbf{r})$, defined in (14), is equal to the ratio of two finite polynomials in z. Hence, $h^{-1}(z; \mathbf{r})$ is a meromorphic function of z, the poles of which follow by solving the equation $h(z; \mathbf{r}) = 0$. The imaginary part of $h(z; \mathbf{r})$ is given by

$$\operatorname{Im} h(z; \mathbf{r}) = \operatorname{Im}(z^2) \left[1 + \sum_{n=1}^{N} \frac{\beta_n^2(\mathbf{r})\omega_n^2(\mathbf{r})}{\rho^2(\mathbf{r})\left|z^2 - \omega_n^2(\mathbf{r})\right|^2} \right]. \tag{A.1}$$

From this result, the inequality

$$|\operatorname{Im} h(z; \mathbf{r})| > |\operatorname{Im}(z^2)| \tag{A.2}$$

is manifest. We see that the square z^2 must be real, otherwise $h(z; \mathbf{r})$ cannot equal zero. For $z^2 \leqslant 0$, the inequality

$$h(ib; \mathbf{r}) < -\omega_0^2(\mathbf{r}) \tag{A.3}$$

comes into play, where b is real. Clearly, $h(z; \mathbf{r})$ can equal zero only if z^2 is real and positive. We confirm our surmise that $h^{-1}(z; \mathbf{r})$ is meromorphic, with all poles lying on the real axis, symmetric with respect to the origin.

The function $h_c(z; \mathbf{r})$, the continuum counterpart of $h(z; \mathbf{r})$, is specified in (63). As long as $\operatorname{Im} z$ differs from zero, the factor of $(\omega^2 - z^2)$ cannot render the denominator of (63) zero. This guarantees that $h_c(z; \mathbf{r})$ is analytic outside the real axis, i.e., on each of its two branches. Since $h_c(z; \mathbf{r})$ is obtained by applying a limiting process to $h(z; \mathbf{r})$, the inequalities (A.2) and (A.3) are valid for $h_c(z; \mathbf{r})$ as well. Therefore, $h_c(z; \mathbf{r})$ does not vanish outside the real axis.

In virtue of (16), the dielectric function $\varepsilon_c(z; \mathbf{r})$ inherits the branch cut of $h_c(z; \mathbf{r})$, and is analytic on each of its two branches. The equality $\varepsilon_c(z; \mathbf{r}) = 0$ implies that $\operatorname{Im} h_c(z; \mathbf{r})$ becomes zero. For $\operatorname{Re} z \neq 0$ this contradicts (A.2). As before, the case $\operatorname{Re} z = 0$ must be checked separately. This can happen via (A.3). Altogether, we conclude that $\varepsilon_c(z; \mathbf{r})$ is analytic and nonzero on each of its two branches.

Before we can make statements about the eigenvalue $\lambda(k, z)$, we have to pay attention to the eigenvector $\mathbf{u}(k, z; \mathbf{r})$, which is the solution of (21) and (22). As is manifest from (17), the dependence of the operator $L(s; \mathbf{r})$ on the real variable s is smooth, except for the points where $h(s; \mathbf{r})$ equals zero. The corresponding divergencies do not appear in $\mathbf{u}(k, s; \mathbf{r})$, owing to the normalization (19). This incites us to suppose that $\mathbf{u}(k, s; \mathbf{r})$ is regular and smooth for all real s. Consequently, $\mathbf{u}(k, z; \mathbf{r})$ is analytic in a certain strip around the real axis. Outside this strip the operator $L(z; \mathbf{r})$ is nowhere singular, so it seems reasonable to assume that $\mathbf{u}(k, z; \mathbf{r})$ is an entire function in the complex z plane. Then the same goes for $[\mathbf{u}(k, z^*; \mathbf{r})]^*$ by the rules of complex conjugation.

The results for $h_c(z; \mathbf{r})$ and $\varepsilon_c(z; \mathbf{r})$, which were proved above, illustrate that the continuum limit does not create any new singularities outside the real axis. It merely modifies the character of the singularities that already exist on the real axis. Poles unite and form a branch cut. We assume that also for $\mathbf{u}(k, z; \mathbf{r})$ the continuum limit does not create any new singularities. Said differently, the eigenvector remains an entire function. Note that all of our assertions on $\mathbf{u}(k, z; \mathbf{r})$ are trivial for the case of a homogeneous dielectric, because then the dependence of $\mathbf{u}(k, z; \mathbf{r})$ on z is absent.

In (25), the eigenvalue $\lambda(k, z)$ is expressed in terms of a spatial integral over the volume V. The above results tell us that the integrand is analytic in the whole complex z plane, except for the real axis. Since V is finite, the integral is convergent if the integrand is bounded. We therefore may conclude that $\lambda(k, z)$ is analytic as long as $\operatorname{Im} z$ differs from zero. The foregoing argument can be repeated for $\lambda_c(k, z)$, so this function is analytic on both of its two branches.

The identity $[\lambda(k, z^*)]^* = \lambda(k, z)$ allows us to put forward

$$\operatorname{Im} \lambda(k, z) = \operatorname{Im} z \left[\frac{\lambda(k, z) - \lambda(k, z^*)}{z - z^*} \right]. \tag{A.4}$$

Let us make $\operatorname{Im} z$ small, and choose $\operatorname{Re} z$ such that $h(z; \mathbf{r})$ is nonzero for $\operatorname{Im} z = 0$. We then avoid divergencies on the real axis. To elaborate (A.4) we invoke the identity

$$\frac{\mathrm{d}\lambda(k, z)}{\mathrm{d}z} \int \mathrm{d}\mathbf{r} [\mathbf{u}(k, z^*; \mathbf{r})]^* \cdot \mathbf{u}(k, z; \mathbf{r}) = \int \mathrm{d}\mathbf{r} [\mathbf{u}(k, z^*; \mathbf{r})]^* \cdot \mathbf{u}(k, z; \mathbf{r}) \frac{\mathrm{d}z^2 \varepsilon(z; \mathbf{r})}{\mathrm{d}z}. \tag{A.5}$$

It can be proved with the help of (25) and partial integration. Before carrying out the replacement $s \to z$ in (25), one has to multiply the left-hand side by the norm $\langle \mathbf{u}(k, s), \mathbf{u}(k, s) \rangle$. After the use of (A.5), the identity (A.4) attains the form

$$\operatorname{Im} \lambda(k, z) \approx \operatorname{Im}(z^2) \left[\int \mathrm{d}\mathbf{r} |\mathbf{u}(k, z; \mathbf{r})|^2 \frac{\mathrm{d}z^2 \varepsilon(z; \mathbf{r})}{\mathrm{d}z^2} \right]_{\operatorname{Im} z = 0} \tag{A.6}$$

where $\operatorname{Im} z$ is small and $h(z; \mathbf{r})$ must be nonzero for $\operatorname{Im} z = 0$.

The result (A.6) invites us to employ the inequality

$$\frac{\mathrm{d}s^2 \varepsilon(s; \mathbf{r})}{\mathrm{d}s^2} > 1 \tag{A.7}$$

which follows from (16). This brings us to

$$|\operatorname{Im} \lambda(k, z)| > |\operatorname{Im}(z^2)| \tag{A.8}$$

where $\operatorname{Im} z$ is small and $\operatorname{Re} z$ is nonzero. Hence, the function $\lambda(k, z)$ surely differs from zero in the vicinity of the real axis. Note that the case $\operatorname{Re} z = 0$ is unimportant, because (21) and (25) force $\lambda(k, z)$ to be real and negative on the imaginary axis. By taking the continuum limit of $\lambda(k, z)$, we do not create zeros for small $\operatorname{Im} z$, because the lower bound (A.8) does not depend on N and Λ. Hence, close to the real axis $\lambda_c(k, z)$ is nonzero.

The above findings on analyticity and location of zeros permit us to make a preliminary statement. We may claim that outside the real axis $\lambda^{-1}(k, z)$ is analytic whenever $\lambda(k, z)$ differs from zero. Therefore, in the vicinity of the real axis $\lambda^{-1}(k, z)$ is analytic. Now the question arises of what happens further in the complex plane, and on the real axis itself.

For the case of a homogeneous dielectric $\lambda(k, z)$ has $(2N + 4)$ zeros and $(2N + 2)$ poles. All of these are located on the real axis, as follows from (A.4). Under a smooth transition from a homogeneous to an inhomogeneous dielectric, the $(4N + 6)$ zeros and poles cannot leave the real axis. This has been demonstrated above. What we now assume is that the transition does not generate any new zeros or any new poles. Then $\lambda^{-1}(k, z)$ is meromorphic, with all poles located on the real axis. Moreover, $\lambda_c(k, z)$ will not possess any zeros on each of its two branches.

To carry out a check on our assumption, we utilize the argument principle [24]. For $\lambda(k, z)$ it can be formulated as

$$\int_{C_2} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}\lambda(k, z)/\mathrm{d}z}{\lambda(k, z)} = (2N + 4) - (2N + 2). \tag{A.9}$$

On the right-hand side the number of poles of $\lambda(k, z)$ is subtracted from the number of zeros of $\lambda(k, z)$. The contour C_2 is a large circle centred around the origin. To prove (A.9) we appeal to (A.5) once more. The ensuing integrand can be simplified with the help of (B.1) and the fact that $\varepsilon(z; \mathbf{r})$ converges to unity for $|z|$ large. Then, (A.9) boils down to the condition $\int_{C_2} \mathrm{d}z/(2\pi\mathrm{i}z) = 1$. This elementary integral completes our consistency check.

Our last job is the analytic continuation of $h_c(z)$. The symmetry (67) allows us to extend the contour of (63) to the negative real axis. By assumption, $\beta^2(z)$ is regular and smooth for all real z, so it must be analytic inside a certain strip around the real axis. Keeping $\mathrm{Im}\, z$ positive as required by (66), we shift in (63) the contour downwards, until it runs below the pole at $\omega = -z$. Upon evaluating the residue we acquire an alternative form of (63), given by

$$h_c(z) = z^2 - \tilde{\omega}_{0,c}^2 + \frac{\mathrm{i}\pi z}{2\rho^2}\beta^2(-z) + \frac{1}{2}\int_{C_4} \mathrm{d}\omega\, \frac{\omega^2\beta^2(\omega)}{\rho^2(\omega^2 - z^2)}. \tag{A.10}$$

The contour C_4 is a straight line, running parallel to the real axis at $\mathrm{Im}\,\omega = -\gamma$, with γ positive.

Rather than the upper half plane, the strip $-\gamma < \mathrm{Im}\, z < \gamma$ constitutes the region where the representation (A.10) is analytic. We thus have succeeded in finding an analytic continuation of (63) below the real axis. From (A.10), we infer the symmetry relation

$$[h_c(-z^*)]^* = h_c(z). \tag{A.11}$$

Note that the operation $z \to -z^*$ maps the strip $-\gamma < \mathrm{Im}\, z < \gamma$ onto itself, so in employing (A.11) domain questions do not arise. In view of (16) and (25), as well as the fact that $\mathbf{u}(k, z)$ is an entire function, the analytic continuation of $\varepsilon_c(\omega + \mathrm{i}\eta)$ and $\lambda_c(k, \omega + \mathrm{i}\eta)$ can be performed on the basis of (A.10).

All properties of h and related functions, which are needed in the main text, have now been proved or made plausible. In essence only one basic assumption on the partial differential equation (22) underlies our discussion. For the eigenvector $\mathbf{u}(k, z; \mathbf{r})$, the eigenvalue $\lambda(k, z)$, as well as its inverse $\lambda^{-1}(k, z)$, both the continuum limit, and the transition from a homogeneous to an inhomogeneous dielectric, do not give birth to singular points out in the complex plane. As a justification, we point out that all singular points of the operator $L(z; \mathbf{r})$ lie on the real axis, regardless of the decision to carry out one of the two aforementioned procedures. Of course, the check (A.9) provides further support for our views.

Appendix B. Verification of two canonical commutation relations

In subsection 3.2, we commenced the verification of the upper two commutators (41). Upon making the transition to complex integration, we arrived at the sufficient conditions (45)–(47). The reflection principle $[\lambda(k, z^*)]^* = \lambda(k, z)$ ensures that the integrals (45) and (47) are

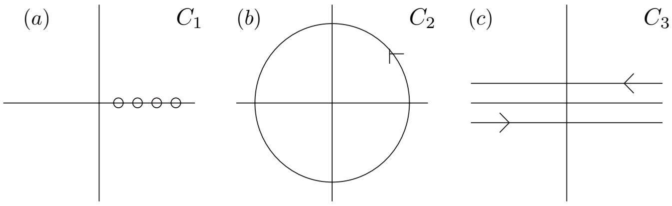

Figure B1. Integration contours in the complex plane z. Horizontal axis: $\mathrm{Im}\, z = 0$; vertical axis: $\mathrm{Re}\, z = 0$. (*a*) Contour C_1 consists of $(2N + 4)$ circles of infinitesimal radius; each circle runs in counterclockwise direction and is centred around a point of the set $\{z = \Omega(k, l)\}$. (*b*) Contour C_2 consists of a single circle of large radius, centred around the origin. (*c*) Contour C_3 consists of two straight lines running above and below the real axis at infinitesimal distance.

invariant under complex conjugation, combined with the interchange of position vectors and tensorial indices. This invariance is imposed by the tensorial delta functions figuring in the right-hand sides of (45) and (47). We first get over with (45), which is the simplest condition.

With the help of (21) and the transformation $z \to -z$, we can include the poles at $z = -\Omega(k, l)$ in the contour. In front of the integral a factor of $1/2$ appears. The new contour C_2 (see figure B1) is a circle $z = R\exp(\mathrm{i}\phi), 0 \leqslant \phi \leqslant 2\pi$. All poles $z = \pm\Omega(k, l)$ lie in the interior of C_2, because R can be taken arbitrarily large. Note that, except for the poles $z = \pm\Omega(k, l)$, the integrand is everywhere analytic. A crucial advantage of C_2 over C_1 is that one can benefit from asymptotic results. If $|z|$ is large, (22) has the algebraic form

$$\frac{z^2}{\lambda(k, z)}\mathbf{u}(k, z; \mathbf{r}) \sim \mathbf{u}(k, z; \mathbf{r}). \tag{B.1}$$

The condition (45) becomes

$$\sum_k \int_{C_2} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z} [\mathbf{u}_T(k, z^*; \mathbf{r}')]^* \mathbf{u}_T(k, z; \mathbf{r}) = \boldsymbol{\delta}_T(\mathbf{r}' - \mathbf{r}). \tag{B.2}$$

The use of (B.1) has modified the analytical structure of the integrand. The poles at $z = \pm\Omega(k, l)$ have given way to a single pole at the origin.

We effectuate a last deformation of our integration contour. Let C_3 enclose the real axis by means of two straight lines running from $+\infty + \mathrm{i}\eta$ to $-\infty + \mathrm{i}\eta$, and from $-\infty - \mathrm{i}\eta$ to $+\infty - \mathrm{i}\eta$, where η is infinitesimally positive. The integrand in (B.2) is still analytic outside the real axis, so integration along C_3 instead of C_2 is permitted. Now we are in a position to appeal to the completeness relation (20). This leaves us with the condition $\int \mathrm{d}z/(2\pi\mathrm{i}z) = 1$, which is indeed true for the contour C_3.

To get to grips with the analytical structure of the integrand of (46), we make the replacement $s \to z$ in (22), and write the result as

$$\frac{\alpha^2(\mathbf{r}) z^2 \mathbf{u}(k, z; \mathbf{r})}{\epsilon_0 \rho(\mathbf{r}) h(z; \mathbf{r}) \lambda(k, z)} = \lambda^{-1}(k, z)\{-c^2 \nabla \times [\nabla \times \mathbf{u}(k, z; \mathbf{r})] + z^2 \mathbf{u}(k, z; \mathbf{r})\} - \mathbf{u}(k, z; \mathbf{r}). \tag{B.3}$$

Obviously, the function on the left-hand side has the same analytic structure as $\lambda^{-1}(k, z)$. Hence, (46) does not really cause any new complications. We first include the regular point $z = 0$ in the contour C_1, and next perform the substitution (B.3). Deformation of contours eventually leads to the condition $\int \mathrm{d}z/[zh(z; \mathbf{r})] = 0$, where C_3 is the contour. As shown in appendix A, all poles of the meromorphic function $h^{-1}(z; \mathbf{r})$ are located on the real axis. After deformation of C_3 to the large circle C_2, the integral indeed disappears, because $h(z; \mathbf{r})$ behaves as z^2 for $|z|$ large.

Since the integrand of (47) has two factors of h in the denominator, its analytic structure must be carefully investigated before any contour deformations can be undertaken. This can be done by adding a new contribution to the integrand, the analytic structure of which is determined by the function $h^{-1}(z;\mathbf{r}')$. Instead of the integrand itself, we consider the sum

$$\mathsf{f}(k,z;\mathbf{r}',\mathbf{r}) = \frac{z^4[\mathbf{u}(k,z^*;\mathbf{r}')]^*\mathbf{u}(k,z;\mathbf{r})}{h(z;\mathbf{r}')h(z;\mathbf{r})\lambda(k,z)} + \frac{z^2\epsilon_0\rho(\mathbf{r})[\mathbf{u}(k,z^*;\mathbf{r}')]^*\mathbf{u}(k,z;\mathbf{r})}{\alpha^2(\mathbf{r})h(z;\mathbf{r}')}. \tag{B.4}$$

The reason is that the poles of the function f can be easily located. Upon employing (B.3) twice, one recognizes that f has the same analytic structure as $\lambda^{-1}(k,z)$. Thus, the integrand of (47) has poles both for $\lambda(k,z)=0$ and for $h(z;\mathbf{r}')=0$. Again, we include the regular point $z=0$ in the contour C_1. Now we can start shifting contours. At each point $z=\Omega(k,l)$ the second contribution to f is regular, so integration along C_1 yields zero. Consequently, in (47) the integrand may be exchanged for f/z. Next, one can replace C_1 by C_2, if the transformation $z\to -z$ is used. Upon employing (B.4) once more, one arrives at the condition

$$\sum_k \int_{C_2} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left[\frac{z^2}{h(z;\mathbf{r})\lambda(k,z)} + \frac{\epsilon_0\rho(\mathbf{r})}{\alpha^2(\mathbf{r})}\right] \frac{z[\mathbf{u}(k,z^*;\mathbf{r}')]^*\mathbf{u}(k,z;\mathbf{r})}{h(z;\mathbf{r}')} = \frac{\epsilon_0\rho(\mathbf{r})}{\alpha^2(\mathbf{r})}\delta(\mathbf{r}'-\mathbf{r}). \tag{B.5}$$

We apply (B.1) to the first term between square brackets. Subsequently, we replace C_2 by C_3, so that (20) comes into play again. Eventually, the integrals $\int \mathrm{d}z\, z/[h(z;\mathbf{r})h(z;\mathbf{r}')]=0$ and $\int \mathrm{d}z\, z/[2\pi\mathrm{i}h(z;\mathbf{r})]=1$ must be proved. As before, this can be done by passing over from the contour C_3 to the large circle C_2.

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